



Families of Exposing Maps in Strictly Pseudoconvex Domains

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Abstract

We prove that given a family (G_t) of strictly pseudoconvex domains varying in \mathcal{C}^2 topology on domains, there exists a continuously varying family of exposing maps $h_{t,\zeta}$ for all G_t at every $\zeta \in \partial G_t$.

Keywords Strictly pseudoconvex domains · Exposing mappings

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1 Introduction

Let $G \subset \subset \mathbb{C}^n$ be a domain and let $\zeta \in \partial G$. We say that ζ is a *globally strongly convex* boundary point of G if ∂G is of class \mathcal{C}^2 and strongly convex at ζ , and $\overline{G} \cap T_\zeta(\partial G) = \{\zeta\}$, where $T_\zeta(\partial G)$ denotes the tangent hyperplane of ∂G at ζ . It is known (cf. [5]) that

Theorem 1.1 *If G is strictly pseudoconvex and has boundary of class \mathcal{C}^2 , then for every $\zeta \in \partial G$ there exist a neighbourhood \hat{G} of \overline{G} and a holomorphic embedding $h : \hat{G} \rightarrow \mathbb{C}^n$ such that $h(\zeta)$ is a globally strongly convex boundary point of $h(G)$.*

Such an h is called an *exposing mapping of G at ζ* . The exposing maps are useful in the investigation of the boundary behaviour of the intrinsic metrics (see [6] or [26]), in the studies on squeezing function (see, for example [4]), and in the proof of the boundary version of the open mapping theorem for holomorphic mappings

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between strictly pseudoconvex domains (see [2]). See also a survey article [27] and the references therein.

A point ζ as above is called a *peak point* with respect to $\mathcal{O}(\overline{G})$, the family of functions holomorphic in a neighbourhood of \overline{G} , if there exists a function $f \in \mathcal{O}(\overline{G})$ such that $f(\zeta) = 1$ and $f(\overline{G} \setminus \{\zeta\}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Such an f is called a *peak function for G at ζ* .

The following question has been formulated in [4]:

Problem 1.2 *Let $\rho : \mathbb{D} \times \mathbb{C}^n \rightarrow \mathbb{R}$ be a plurisubharmonic function of class \mathcal{C}^k , $k \in \mathbb{N}$, $k \geq 2$. Assume that for any $t \in \mathbb{D}$ the truncated function $\rho|_{\{t\} \times \mathbb{C}^n}$ is strictly plurisubharmonic and globally defines a bounded strictly pseudoconvex domain $G_t := \{w \in \mathbb{C}^n : \rho(t, w) < 0\}$. This latter can be understood as a family of strictly pseudoconvex domains with boundaries of class \mathcal{C}^k over \mathbb{D} . Do there exist \mathcal{C}^{k-2} -continuously varying families:*

- (A) $(f_{t,\zeta})_{t \in \mathbb{D}, \zeta \in \partial G_t}$ of peak functions for G_t at $\zeta \in \partial G_t$.
- (B) $(h_{t,\zeta})_{t \in \mathbb{D}, \zeta \in \partial G_t}$ of exposing maps for G_t at $\zeta \in \partial G_t$?

In the papers [15] and [16] we have affirmatively answered the question (A). In [15] we treated the particular case, where the parameter space \mathbb{D} was replaced with some compact metric space, and the constructed family of peak functions was continuous with respect to the parameter (actually, it was continuous with respect to all variables). Later, in [16], we considered the problem (A) in its full generality. The hereby paper is, in the author's intention, parallel to [15] for the problem (B): we show that, under some additional assumption, given a family of domains G_t as in Problem 1.2, for any compact $K \subset \mathbb{D}$, there exists a continuous family $(h_{t,\zeta})_{t \in K, \zeta \in \partial G_t}$ of exposing maps for G_t at $\zeta \in \partial G_t$, $t \in K$. Namely, we prove

Theorem 1.3 *Let $(G_t)_{t \in \mathbb{D}}$ be a family of strictly pseudoconvex domains as in Problem 1.2 with $k = 2$. Let $\sigma \in (0, 1)$. Take an $R > 0$ such that $\bigcup_{t \in \sigma \overline{\mathbb{D}}} \overline{G}_t \subset \subset \mathbb{B}(0, R)$. Assume that there exist a \mathcal{C}^2 -continuous family $(\gamma_{t,\zeta})_{t \in \sigma \overline{\mathbb{D}}, \zeta \in \partial G_t}$ of smooth embedded arcs $[0, 1] \rightarrow \mathbb{C}^n$ such that $\gamma_{t,\zeta}(0) = \zeta$, $\gamma_{t,\zeta}(1) \in \mathbb{S}^{2n-1}(R)$ and $\gamma_{t,\zeta}(x) \in \mathbb{C}^n \setminus (\overline{G}_t \cup \mathbb{S}^{2n-1}(R))$, $x \in (0, 1)$, for all $t \in \sigma \overline{\mathbb{D}}$ and $\zeta \in \partial G_t$. Then there exists a family $(h_{t,\zeta})_{t \in \sigma \overline{\mathbb{D}}, \zeta \in \partial G_t}$ of exposing maps for G_t at ζ , continuous with respect to all variables.*

Here and below $\mathbb{B}(a, R)$ stands for the open ball in \mathbb{C}^n with centre at a and radius $R > 0$, and $\mathbb{S}^{2n-1}(R) := \partial \mathbb{B}(0, R)$.

Remark 1.4 Our assumption concerning the \mathcal{C}^2 -continuity of the family $(\gamma_{t,\zeta})_{t \in \sigma \overline{\mathbb{D}}, \zeta \in \partial G_t}$ should be understood in the following way:

For each t let Γ_t be a neighbourhood of ∂G_t with $\nabla r_t \neq 0$ on Γ_t , where $r_t := \rho(t, \cdot)$ and ∇r_t denotes its gradient. The neighbourhoods Γ_t may be chosen to depend in a \mathcal{C}^2 -continuous way on t .

Then there exist a positive constants $\sigma' \in (\sigma, 1)$ and $\tilde{\varepsilon}$

such that the family $(\gamma_{t,\zeta})_{t \in \sigma \mathbb{D}, \zeta \in \partial G_t}$ may be extended to a \mathcal{C}^2 -continuous family

$$(\gamma_{t,\zeta})_{t \in \sigma' \mathbb{D}, \zeta \in \bigcup_{|\kappa| < \tilde{\varepsilon}} \partial G_t^{(\kappa)}}$$

of smooth embedded arcs $[0, 1] \rightarrow \mathbb{C}^n$ such that $\gamma_{t,\zeta}(0) = \zeta$, $\gamma_{t,\zeta}(1) \in \mathbb{S}^{2n-1}(R)$ and $\gamma_{t,\zeta}(x) \in \mathbb{C}^n \setminus (\overline{G_t^{(\kappa)}} \cup \mathbb{S}^{2n-1}(R))$, $x \in (0, 1)$, for all $t \in \sigma' \mathbb{D}$ and $\zeta \in \partial G_t^{(\kappa)}$, $|\kappa| < \tilde{\varepsilon}$. Here, for small $|\kappa|$ we have put

$$G_t^{(\kappa)} := (G_t \setminus \Gamma_t) \cup \{z \in \Gamma_t : r_t(z) < \kappa\}.$$

Notice that the assumption concerning the existence of the family $(\gamma_{t,\zeta})$ of suitable embedded arcs is completely in the spirit of Theorem 1.3 from [3], which is a version of our result for a single domain. This kind of assumption is not present in Theorem 1.1, which is indeed a “pointwise” result for single domain. It seems that the existing methods do not allow to relax this additional assumption, with the main obstruction being of rather topological nature. On the other hand, in certain subclasses of the class of strictly pseudoconvex domains, the existence of such family of embedded arcs need not be assumed: in Example 5.1 we show that if the domains G_t are all strongly linearly convex, then the family $(\gamma_{t,\zeta})$ can always be constructed and therefore the family $(h_{t,\zeta})_{t \in \sigma \mathbb{D}, \zeta \in \partial G_t}$ always exist. This latter should be compared with Theorem 1.4 from [5], which says that if a single domain $G \subset \subset \mathbb{C}^n$ is convex, smoothly bounded, and of finite type $2l$ ($l \in \mathbb{N}$), then there exists a *smooth* family $(h_\zeta)_{\zeta \in \partial G}$ of exposing maps for G at ζ (moreover, in such a case, each h_ζ may be chosen to be a holomorphic automorphism of \mathbb{C}^n), and with Theorems 5.1 and 5.2 from [3].

We propose the proof of Theorem 1.3, which merges the methods of [5] with those from [3]. In the first part of the proof, ideologically similar to Lemma 3.1 from [5], we deliver some parametric version of Narasimhan lemma (see [9] for another approach), thus constructing the family of local and locally exposing maps. Then, with the aid of [24], we find the family of global and locally exposing maps. In the final part of the proof, based on ideas from [3], we pass to the construction of the required family of exposing maps. The main tools will be the parametric version of the so-called Forstnerič splitting lemma for biholomorphic maps due to Simon (see Theorem 2.4) and the following parametric version of higher-dimensional Mergelyan approximation theorem [7, Theorem 21]. Although some authors refer to certain parametric versions of Mergelyan theorem (cf. [5]), we were not able to find any in the literature. Also, versions referred in mentioned sources seem to be not suitable for our purposes.

Theorem 1.5 *Let $S = K \cup M \subset \mathbb{C}^n$ be admissible in the sense of [7], i.e. S and K are a Stein compacts and $M = S \setminus K$ is a totally real submanifold of class \mathcal{C}^k (with boundary) with some $k \in \mathbb{N}$, let $(g_t)_{t \in T} \subset \mathcal{C}^k(W) \cap \mathcal{O}(V)$ be a family of functions continuously dependent on all variables together with the parameter $t \in T$, where T is a compact metric space and V, W are some open neighbourhoods of K, S , respectively. Then there exists an open neighbourhood Ω of S such that for any $\varepsilon > 0$ there exists $(f_t)_{t \in T} \subset \mathcal{O}(\Omega)$, a family of functions continuously dependent on all variables and such that for all $t \in T$ we have $\|g_t - f_t\|_{\mathcal{C}^k(S)} < \varepsilon$.*

Theorem 1.5 is proved in Sect. 3, while the proof of our main result, Theorem 1.3, is presented in Sect. 4. Finally, in Sect. 5, we discuss the case of the family of strongly linearly convex domains. We start with some preliminary results, presented in Sect. 2, and end with some concluding remarks in Sect. 6.

2 Preliminaries

Definition 2.1 Let $G \subset \subset \mathbb{C}^n$ be a domain. It is called a *strictly pseudoconvex* if there exist a neighbourhood U of ∂G and a *defining function* $r : U \rightarrow \mathbb{R}$ of class \mathcal{C}^2 on U and such that

- (i) $G \cap U = \{z \in U : r(z) < 0\}$,
- (ii) $(\mathbb{C}^n \setminus \overline{G}) \cap U = \{z \in U : r(z) > 0\}$,
- (iii) $\nabla r(z) \neq 0$ for $z \in \partial D$, where $\nabla r(z) := \left(\frac{\partial r}{\partial z_1}(z), \dots, \frac{\partial r}{\partial z_n}(z) \right)$,

together with

$$\mathcal{L}_r(z; X) > 0 \text{ for } z \in \partial G \text{ and nonzero } X \in T_z^{\mathbb{C}}(\partial G),$$

where \mathcal{L}_r denotes the Levi form of r and $T_z^{\mathbb{C}}(\partial G)$ is the complex tangent space to ∂G at z .

It is known that U and r can be chosen to satisfy (i)-(iii) and, additionally:

- (iv) $\mathcal{L}_r(z; X) > 0$ for $z \in U$ and all nonzero $X \in \mathbb{C}^n$,

cf. [13].

Note that for a function r as above and a point $\zeta \in \partial G$, the Taylor expansion of r at ζ has the following form:

$$r(z) = r(\zeta) - 2\operatorname{Re} P_r(z; \zeta) + \mathcal{L}_r(\zeta; z - \zeta) + o(\|z - \zeta\|^2), \quad (2.1)$$

where

$$P_r(z; \zeta) := - \sum_{j=1}^n \frac{\partial r}{\partial z_j}(\zeta)(z_j - \zeta_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 r}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j)$$

is the *Levi polynomial* of r at ζ .

In Sect. 5, we shall discuss the stronger notion than that of strictly pseudoconvex domains. Namely, we need the following

Definition 2.2 A domain $G \subset \subset \mathbb{C}^n$ with \mathcal{C}^2 boundary is called *strongly linearly convex* if there exists a defining function r for G with

$$\mathcal{L}_r(z; X) > \left| \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial z_k}(z) X_j X_k \right|,$$

for all $z \in \partial G$ and all nonzero $X = (X_1, \dots, X_n) \in T_z^{\mathbb{C}}(\partial G)$.

Finally, an important ingredient of the proof of Theorem 1.3 is the parametric version of Forstnerič's splitting lemma for biholomorphic maps close to identity.

Definition 2.3 A pair (A, B) of compact subsets of \mathbb{C}^n is called a *Cartan pair*, if

- (i) $A, B, A \cup B, A \cap B$ are all Stein compacts,
- (ii) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.

The following comes from [21] (see also [20], and [8] for a nonparameter version). Henceforth for a set $Z \subset \mathbb{C}^n$ and a number $\delta > 0$ we abbreviate $Z^\delta := \bigcup_{z \in Z} \mathbb{B}(z, \delta)$.

Theorem 2.4 Let \mathcal{T} be a nonempty compact topological space, let $(A_t, B_t)_{t \in \mathcal{T}}$ be an admissible (in the sense of [21]) family of Cartan pairs, and let $\mu > 0$. Then there exists a $\tau > 0$ such that for any $\eta > 0$ there exists an $\varepsilon_\eta > 0$ with the property that for any family $(\gamma_t : (A_t \cap B_t)^\mu \rightarrow \mathbb{C}^n)_{t \in \mathcal{T}}$ of injective holomorphic maps satisfying $\|\gamma_t - \text{Id}\|_{(A_t \cap B_t)^\mu} < \varepsilon_\eta$, $t \in \mathcal{T}$ and depending continuously with respect to all variables, there exist families $(\alpha_t : A_t^{2\tau} \rightarrow \mathbb{C}^n)_{t \in \mathcal{T}}$, $(\beta_t : B_t^{2\tau} \rightarrow \mathbb{C}^n)_{t \in \mathcal{T}}$ of injective holomorphic maps, depending continuously on all variables, and such that for all $t \in \mathcal{T}$ we have

- (1) $\gamma_t = \beta_t \circ \alpha_t^{-1}$ on $(A_t \cap B_t)^\tau$, and
- (2) $\|\alpha_t - \text{Id}\|_{(A_t)^{2\tau}} < \eta$, $\|\beta_t - \text{Id}\|_{(B_t)^{2\tau}} < \eta$.

Remark 2.5 In the proof of Theorem 1.3 we shall apply Theorem 2.4 in the situation where for sufficiently small positive k

$$A_{t,\zeta} = \overline{D_{t,\zeta}} \cap \overline{\mathbb{B}}(\zeta, k), \quad B_{t,\zeta} = \overline{D_{t,\zeta}} \setminus \mathbb{B}\left(\zeta, \frac{k}{2}\right).$$

with the indices (t, ζ) taken from a compact set of an Euclidean space and $(D_{t,\zeta})$ forming a family of strictly pseudoconvex domains varying in a \mathcal{C}^2 -continuous manner, and with the property that $\zeta \in \partial D_{t,\zeta}$. In this case, it is possible to choose $(\alpha_{t,\zeta})$ and $(\beta_{t,\zeta})$ as above, where the functions $\alpha_{t,\zeta}$ additionally interpolate identity at ζ to an arbitrarily high order, cf. remarks from Lemma 5.3 in [3] and from Lemma 5.2 in [5].

3 Proof of Theorem 1.5

The proof of Theorem 1.5 goes along the lines of the proof of Theorem 21 from [7], with the most important modification at the end, where we use methods of [17], together with the parameter dependence of the solution of the 1st Cousin Problem with bounds (to be deduced from the proof of Theorem VII.6.3 in [19]). We include the proof here for the convenience of the reader and in order to be able to point out the modifications of it needed in getting the target described in Remark 3.1 below.

Proof Step 1: Suppose that there exists a $t \in T$ such that $\text{spt} g_t \cap K \neq \emptyset$.

Take $\tilde{\Omega}$, a Stein neighbourhood of S such that $K_0 := \hat{K}_{\mathcal{O}(\tilde{\Omega})} \subset V$ (cf. [7, Lemma 2]). Let $K_1 \subset V$ be a $\mathcal{O}(\tilde{\Omega})$ -convex compact set such that $K_0 \subset \text{int} K_1$. Choose χ , a smooth cutoff function with support equal K_1 and with the property that $\chi = 1$ in a neighbourhood of K_0 . By Oka–Weil theorem with parameters ([8, Theorem 2.8.1]), there exists a continuous family $(\varphi_t)_{t \in T} \subset \mathcal{O}(\tilde{\Omega})$ with

$$\sup_{z \in K_1, t \in T} |g_t(z) - \varphi_t(z)| < \varepsilon.$$

Let

$$\tilde{g}_t := \chi \varphi_t + (1 - \chi)g_t = \varphi_t + (1 - \chi)(g_t - \varphi_t).$$

Then $\|\tilde{g}_t - g_t\|_{\mathcal{C}^k(S)} < C\varepsilon$ for all $t \in T$ with positive constant C depending only on K and χ . Therefore, it remains to show that we are able to approximate the family $(\psi_t := (1 - \chi)(g_t - \varphi_t))_{t \in T} \subset \mathcal{C}^k(S)$, which enjoys the property that the supports of the functions ψ_t do not intersect some (common) neighbourhood of K_0 . We therefore can pass to the following:

Step 2: Approximation of $(\psi_t)_{t \in T}$ (if the condition in Step 1 is empty, we go directly to Step 2).

If for some $t \in T$ we have $\text{spt} \psi_t \cap \partial M \neq \emptyset$, we take a bigger totally real submanifold, still denoted by M , of class \mathcal{C}^k , by extending M through ∂M , and we extend all functions $\psi_t|_M$ to the functions of class \mathcal{C}^k with compact supports contained in the relative interior of M (we keep the notation ψ_t for these extensions). Note that this extension of M may be taken one good for all $t \in T$ and the extensions of functions ψ_t may be taken to depend continuously on all variables. Using now the fact that χ above equals 1 on a neighbourhood, say V_0 , of K_0 , we may multiply everything by another cutoff function, obtaining the existence of some compact set in $F \subset M \setminus K$ such that for all $t \in T$ we have $\text{spt} \psi_t|_M \subset F$.

We cover F by a finite number of domains $M_1, \dots, M_m \subset M$ such that for every $j \in \{1, \dots, m\}$ Proposition 2 from [7] (one may also bear in mind Proposition at the beginning of Section 3 from [17]) holds true for M_j (observe that the functions f_ε constructed therein changes continuously if the input data are perturbed in a continuous way) and $\bigcup_{j=1}^m \overline{M}_j \subset M \setminus K$. Let $(\chi_j \in \mathcal{C}_0^k(M_j))_{j=1}^m$ be a partition of unity subordinated to the cover M_1, \dots, M_m of a neighbourhood of F so that for all $t \in T$ we have $\psi_t = \sum_{j=1}^m \chi_j \psi_t$. We see it suffices to approximate every single family $(\chi_j \psi_t)_{t \in T}$ with j fixed. Without loss of generality, assume that $\text{spt} \psi_t \subset \subset M_1$, $t \in T$. Let $U \subset \mathbb{C}^n$ be a neighbourhood of ∂M_1 as in (b) from Proposition 2 in [7] (observe it is independent of ψ_t). Take open sets $A, B \subset \mathbb{C}^n$ such that

$$M_1 \subset B, \quad S \setminus M_1 \subset A, \quad A \cap B \subset U.$$

Let Ω be a Stein neighbourhood of S with $\Omega \subset (A \cup B) \cap \tilde{\Omega}$ and define

$$\Omega_A := \Omega \cap A, \quad \Omega_B := \Omega \cap B.$$

By shrinking Ω little bit, we may assume it is a strictly pseudoconvex domain with smooth boundary. Now, analysing the proof of Theorem VII.6.3 from [19], we see the solution of the 1st. Cousin Problem with bounds depends continuously on parameters if only input functions are entire and taken to also depend continuously on parameters. Now, if we consider the family $(h_t)_{t \in T} \subset \mathcal{O}(\mathbb{C}^n)$ of functions, continuously dependent on all variables, given by Proposition 2 from [7], with

$$\|\psi_t - h_t\|_{C^k(M_1)} < \varepsilon \text{ and } \|h_t\|_U < \varepsilon, \quad t \in T,$$

we find, by above observation, the continuous (in all variables) families of functions $(h_t^A)_{t \in T} \in \mathcal{O}(\Omega_A)$ and $(h_t^B)_{t \in T} \in \mathcal{O}(\Omega_B)$ such that for a positive constant D , independent of $t \in T$, we have

$$\|h_t^A\|_{\Omega_A} < D\varepsilon, \quad \|h_t^B\|_{\Omega_B} < D\varepsilon,$$

and

$$h_t = h_t^A - h_t^B$$

on $\Omega \cap A \cap B$ for all $t \in T$. Finally we put $f_t := h_t + h_t^B$ on Ω_B and $f_t := h_t^A$ on Ω_A . Invoking the Cauchy estimates, we get the conclusion. \square

Remark 3.1 Actually, in the proof of Theorem 1.3 we shall need stronger version of Theorem 1.5, where the domains of definition of certain *injections* g_t will vary, the functions g_t themselves will depend in a C^2 -continuous way on all variables and we will need to construct a family f_t of holomorphic *embeddings*, depending in a C^2 -continuous way on all variables and admitting an interpolation to order 3 at a certain point. We shall include the details within the proof of Theorem 1.3.

4 Proof of Theorem 1.3

Proof of Theorem 1.3 Let $r_t := \rho(t, \cdot)$, $t \in \mathbb{D}$. As in Remark 1.1, for each t let Γ_t be a neighbourhood of ∂G_t , C^2 -continuously dependent on t , and with $\nabla r_t \neq 0$ on Γ_t . Take positive $\sigma' \in (\sigma, 1)$ and $\tilde{\varepsilon}$ such that the family $(\gamma_{t,\zeta})_{t \in \sigma\overline{\mathbb{D}}, \zeta \in \partial G_t}$ may be extended to a C^2 -continuous family

$$(\gamma_{t,\zeta})_{t \in \sigma'\overline{\mathbb{D}}, \zeta \in \bigcup_{|\kappa| < \tilde{\varepsilon}(t)} \partial G_t^{(\kappa)}}$$

of smooth embedded arcs $[0, 1] \rightarrow \mathbb{C}^n$ such that $\gamma_{t,\zeta}(0) = \zeta$, $\gamma_{t,\zeta}(1) \in \mathbb{S}^{2-1}(R)$ and $\gamma_{t,\zeta}(x) \in \mathbb{C}^n \setminus (\overline{G_t^{(\kappa)}} \cup \mathbb{S}^{2-1}(R))$, $x \in (0, 1)$, for all $t \in \mathbb{D}$ and $\zeta \in \partial G_t^{(\kappa)}$, $|\kappa| < \tilde{\varepsilon}$.

After possible decreasing of $\tilde{\varepsilon}$, there exists a C^2 -continuous family

$$(l_{t,\zeta})_{t \in \mathbb{D}, \zeta \in \bigcup_{|\kappa| < \tilde{\varepsilon}} \partial G_t^{(\kappa)}}$$

of global changes of coordinates, being compositions of translation and unitary transformation, such that $l_{t,\zeta}(\zeta) = 0$ and with the property that $\mathbf{n}_{t,\zeta}$, a unit exterior normal vector to $\partial G_t^{(\kappa)}$ at ζ is transformed to a vector $(1, 0, \dots, 0)$. In particular, in these new coordinates we have $T_\zeta^{\mathbb{C}}(\partial G_t^{(\kappa)}) = \{w_1 = 0\}$. Write $l_{t,\zeta}(z) = \Phi_{t,\zeta}(z - \zeta)$ with $\Phi_{t,\zeta}$ being a unitary matrix depending \mathcal{C}^2 -continuously on $t \in \mathbb{D}$ and $\zeta \in \partial G_t^{(\kappa)}$.

Denote by $P_{t,\zeta}$ the Levi polynomial of r_t at ζ . It is standard that there exist positive constants C, ξ , and λ such that for any $t \in \sigma'\mathbb{D}$, any $\zeta \in \Gamma_t$ with $\text{dist}(\zeta, \partial G_t) < \xi$, and any $z \in \mathbb{B}(\zeta, \lambda)$ we have

$$r_t(z) \geq r_t(\zeta) - 2\text{Re} P_{t,\zeta}(z) + C\|z - \zeta\|^2.$$

In particular, if $\zeta \in \partial G_t$ and $z \in \overline{G}_t$ is such that $\|z - \zeta\| < \lambda$, we have

$$-\text{Re} P_{t,\zeta}(z) \leq -\frac{C}{2}\|\zeta - z\|^2.$$

Putting $\hat{P}_{t,\zeta}(z) := P_{t,\zeta}(l_{t,\zeta}^{-1}(z))$ we get the estimate

$$-\text{Re} \hat{P}_{t,\zeta}(z) \leq -\frac{C}{2}\|\zeta - l_{t,\zeta}^{-1}(z)\|^2 = -\frac{C}{2}\|\Phi_{t,\zeta}^{-1}z\|^2 = -\frac{C}{2}\|z\|^2 \quad (4.1)$$

for $z \in l_{t,\zeta}(G_t) \cap \mathbb{B}(0, p)$ (and all $t \in \sigma'\mathbb{D}$ and $\zeta \in \partial G_t$) with some positive p . Consider the mapping

$$\Psi_{t,\zeta}(z) := (-\hat{P}_{t,\zeta}(z), z_2, \dots, z_n) = (-P_{t,\zeta}(l_{t,\zeta}^{-1}(z)), z_2, \dots, z_n),$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We have $\Psi_{t,\zeta}(0) = 0$, $\Psi_{t,\zeta}$ is injective and holomorphic in $\mathbb{B}(0, \rho)$ with some positive ρ (independent of $t \in \sigma'\mathbb{D}$, $\zeta \in \partial G_t$). Also $\mathbf{n}_{t,\zeta}$ becomes $(1, 0, \dots, 0)$ in the local coordinates near ζ given by $\Psi_{t,\zeta} \circ l_{t,\zeta}$.

Define

$$\Omega_{t,\zeta} := l_{t,\zeta}(G_t).$$

Recall that the elements of $\Psi_{t,\zeta}(\Omega_{t,\zeta} \cap \mathbb{B}(0, \rho))$ are of the form $(-\hat{P}_{t,\zeta}(z), z_2, \dots, z_n)$ for $z \in \Omega_{t,\zeta} \cap \mathbb{B}(0, \rho)$.

For $z \in \Omega_{t,\zeta} \cap \mathbb{B}(0, \rho)$, writing $z_1 = x_1 + ix_2$ and $z' = (z_2, \dots, z_n)$, we get from (4.1) the following estimate

$$-\text{Re} \hat{P}_{t,\zeta}(z) - \omega_{t,\zeta}(-\text{Im} \hat{P}_{t,\zeta}(z), z') \leq 0,$$

where

$$\omega_{t,\zeta}(x_2, z') := -D(x_2^2 + \|z'\|^2)$$

with some positive constant D , independent of $t \in \sigma'\mathbb{D}$ and $\zeta \in \partial G_t$. In particular,

$$\overline{\Psi_{t,\zeta}(\Omega_{t,\zeta} \cap \mathbb{B}(0, \rho))} \subset \{z \in \mathbb{C}^n : x_1 - \omega_{t,\zeta}(x_2, z') \leq 0\}, \quad (4.2)$$

which implies that near 0 the domain $\Psi_{t,\zeta}(\Omega_{t,\zeta})$ is strictly 2-convex in the sense of [5].

All the above constructions remain valid if we allow ζ not only from ∂G_t , but also from $\partial G_t^{(\kappa)}$, where $|\kappa| \leq \varepsilon$, with some positive constant $\varepsilon < \tilde{\varepsilon}$ which is taken one good for all $t \in \sigma'\mathbb{D}$. Let us denote from now on

$$N(\partial G_t) := \bigcup_{|\kappa| < \frac{\varepsilon}{2}} \partial G_t^{(\kappa)}, \quad t \in \sigma'\mathbb{D}.$$

Observe that for fixed t and $\zeta \in N(\partial G_t)$ there exists only one κ with $\zeta \in \partial G_t^{(\kappa)}$. Therefore, fixing a pair (t, ζ) actually determines the triple (t, ζ, κ) . We shall frequently use this fact in the sequel without additional comments: namely, we shall only sometimes—when it will be not clear from the context—write $\kappa(t, \zeta)$ to indicate the fact that the particular κ arises from the choice of (t, ζ) . Otherwise, we shall omit this indexing.

Analysing the proof of Proposition 1.2 in [24], we see that there exists a \mathcal{C}^2 -continuous family $(\Theta_{t,\zeta})_{t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)}$ of holomorphic automorphisms of \mathbb{C}^n , that can be represented as

$$\Theta_{t,\zeta}(z) = \Psi_{t,\zeta}(z) + f_{t,\zeta}(z),$$

where $f_{t,\zeta}$ is entire, and there exist positive constants θ and A , independent of $t \in \sigma'\mathbb{D}$ and $\zeta \in N(\partial G_t)$ with the property that for $\|z\| \leq \theta$ we have $\|f_{t,\zeta}(z)\| \leq A\|z\|^3$ (note that the set of parameters in [24] is assumed to be a Stein space. On the other hand, the dependence of parameters considered there is holomorphic. We need only a \mathcal{C}^2 -continuous dependence of parameters and in this case the assumption about Steinness of the set of parameters can be omitted. Also, the very same conclusion is possible to be obtained by using [25]).

For $t \in \sigma'\mathbb{D}$ and $\zeta \in \partial G_t^{(\kappa)}$ put $D_{t,\zeta}^{(\kappa)} := \Theta_{t,\zeta}(l_{t,\zeta}(G_t^{(\kappa)}))$ and observe that for these domains we have $T_0(\partial D_{t,\zeta}^{(\kappa)})$, the real tangent space to $\partial D_{t,\zeta}^{(\kappa)}$ at 0, equals $\{x_1 = 0\}$, $\mathbf{n}_0 = (1, 0, \dots, 0)$, and $\partial D_{t,\zeta}^{(\kappa)}$ is 2-convex near 0 in the sense of (4.2), after possible decreasing ρ and D there, so that they remain to be independent of $t \in \sigma'\mathbb{D}$, $|\kappa| \leq \varepsilon$ after possible decreasing ε , and $\zeta \in \partial G_t^{(\kappa)}$.

As in [3], we may without loss of generality (after possible slight decreasing of σ') modify the family of curves $(\gamma_{t,\zeta})_{t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)}$ so that the initial parts (of uniform length) of the curves $\Theta_{t,\zeta}(l_{t,\zeta}(\gamma_{t,\zeta}))$ are all equal to the segment $e_1[0, s]$ with some positive s , arbitrarily small, and so that they are perpendicular to $\Theta_{t,\zeta}(l_{t,\zeta}(\mathbb{S}^{2n-1}(R)))$ where they intersect. The modification may be carried out so that the modified family, still denoted by $(\gamma_{t,\zeta})_{t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)}$, keeps its regularity.

For any $t \in \sigma'\mathbb{D}$ and any $\zeta \in N(\partial G_t)$ (after possible decreasing ε) let $U_{t,\zeta}$ be a neighbourhood of $\overline{D_{t,\zeta}^{(\kappa)}} \cup (e_1[0, s])$ and $V_{t,\zeta} \subset U_{t,\zeta}$ be a neighbourhood of $D_{t,\zeta}^{(\kappa)}$, both \mathcal{C}^2 -continuously dependent on (t, ζ) , and let a \mathcal{C}^2 -continuous family $(g_{t,\zeta} : U_{t,\zeta} \rightarrow \mathbb{C}^n)_{t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)}$ of smooth embeddings such that $g_{t,\zeta} = \text{Id}$ in $V_{t,\zeta}$, $g_{t,\zeta}$ stretches $e_1[0, s]$ to cover $\Theta_{t,\zeta}(l_{t,\zeta}(\gamma_{t,\zeta}([0, 1])))$, and $g_{t,\zeta}(e_1[0, s])$ is perpendicular to $(\Theta_{t,\zeta} \circ l_{t,\zeta})(\mathbb{S}^{2n-1}(R))$. Note that $U_{t,\zeta}$ and $V_{t,\zeta}$ may be chosen to be independent of $|\kappa| < \varepsilon$, and thus of ζ (after eventually decreasing ε).

We want to apply Theorem 1.5 for small $\varepsilon_0 > 0$ and the family $(g_{t,\zeta})_{(t,\zeta) \in P}$, where P is some relatively compact subset of the open set $\{(t, \zeta) : t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)\}$ containing in its interior all the couples (t, ζ) with $t \in \sigma\overline{\mathbb{D}}, \zeta \in \partial G_t$. In our concrete situation we want to modify the construction of $(f_{t,\zeta})$ therein, taking into account the variable domains of the functions $g_{t,\zeta}$ and in order to get its \mathcal{C}^2 -continuous dependence on all variables and thus allowing the interpolation to order 3 at se_1 with the continuity of new approximating and interpolating family of functions with respect to all variables. Of course the domains of such functions will also depend on t in a suitable way. Moreover, we need to make sure that the functions $f_{t,\zeta}$ are injections in some suitable chosen neighbourhoods of $\overline{D_{t,\zeta}^{(\kappa)}} \cup (e_1[0, s])$ for $(t, \zeta) \in P$. Below we indicate the modifications of the construction of the family $(f_{t,\zeta})$ required when proving a variant of Theorem 1.5 in this particular case.

Variable domains, the \mathcal{C}^2 dependence on all variables and the interpolation at se_1 . Fix P , a relatively compact subset of the open set $\{(t, \zeta) : t \in \sigma'\mathbb{D}, \zeta \in N(\partial G_t)\}$ containing in its interior all the couples (t, ζ) with $t \in \sigma\overline{\mathbb{D}}, \zeta \in \partial G_t$. There exists a small positive constant $\beta < \frac{s}{3}$ with the property that for all couples $(t, \zeta) \in P$ we may find the strictly pseudoconvex domains $H_{t,\zeta}$ with \mathcal{C}^2 -smooth boundaries, depending in a \mathcal{C}^2 -continuous way on (t, ζ) , and satisfying

- (1) $\overline{D_{t,\zeta}^{(\kappa)}} \subset\subset H_{t,\zeta} \subset\subset V_{t,\zeta}$
- (2) $\text{dist}(\overline{D_{t,\zeta}^{(\kappa)}}, \partial H_{t,\zeta}) = \beta$
- (3) $\text{dist}(\overline{H_{t,\zeta}}, \partial V_{t,\zeta}) \geq \beta$
- (4) tube with radius $\frac{\beta}{4}$ around $e_1[0, s]$ is compactly contained in $U_{t,\zeta}$
- (5) $\overline{H_{t,\zeta}} \cap (e_1[\beta, s]) = \emptyset$ and $S_{t,\zeta} := \overline{H_{t,\zeta}} \cup (e_1[\beta, s])$ satisfies the assumptions of Theorem 1.5.

In order to apply the standard interpolation corrections with the continuity of corrected family of functions with respect to all variables, we need to adjust the construction of $(f_{t,\zeta})$ in Theorem 1.5 (for $S = S_{t,\zeta}$) so that we take care of the variable domains $U_{t,\zeta}$ and that it will depend in a \mathcal{C}^2 -continuous way on all variables (as the input data $(g_{t,\zeta})$ do). To get this aim, observe that in our particular situation, in Step 1 of proof of Theorem 1.5 we may take $\varphi_{t,\zeta} = \text{Id}$, $(t, \zeta) \in P$ (and a suitable family $(\chi_{t,\zeta})_{(t,\zeta) \in P}$ of cutoff functions, smoothly dependent on all variables, with supports contained in $V_{t,\zeta}$ and equal one on neighbourhoods of $\overline{H_{t,\zeta}}$ with—by the compactness argument—distances to the boundaries uniformly bounded from below). Putting now $\psi_{t,\zeta} := (1 - \chi_{t,\zeta})(g_{t,\zeta} - \text{Id})_{(t,\zeta) \in P}$, we observe that there exist an $\tilde{s} > s$ and a compact set $F \subset e_1(\beta, \tilde{s}]$ such that for all $(t, \zeta) \in P$ we have $\text{spt} \psi_{t,\zeta}|_{e_1[\beta, \tilde{s}]} \subset F$

(after possible multiplying by suitable cutoff function) and it suffices to approximate the family $(\psi_{t,\zeta})_{(t,\zeta) \in P}$. Therefore, we only have to modify the construction from Step 2 in order to get the better regularity we are after. Here, observe that the proof remains unchanged until we have to choose the open sets $M_1 \subset B$ and $S_{t,\zeta} \setminus M_1 \subset A$ with $A \cap B \subset U$ for all $(t, \zeta) \in P$. Observe that, by the compactness argument, these sets may be chosen to be independent of $(t, \zeta) \in P$.

For a fixed $(t, \zeta) \in P$ one may choose a strictly pseudoconvex and smoothly bounded domains $N_{t,\zeta}$ and $\hat{N}_{t,\zeta}$ such that

$$S_{t,\zeta} \subset\subset N_{t,\zeta} \subset\subset \hat{N}_{t,\zeta} \subset A \cup B,$$

and by Theorem V.2.5 from [19], there exist neighbourhoods

$$\overline{N_{t,\zeta}} \subset V_0^{t,\zeta} \subset\subset V^{t,\zeta} \subset\subset \hat{N}_{t,\zeta}$$

and a solution operator for $\bar{\partial}$ -problem

$$T_1^{V^{t,\zeta}, V_0^{t,\zeta}} : \mathcal{C}_{0,1}(\overline{V^{t,\zeta}}) \rightarrow \mathcal{C}_{0,0}(V_0^{t,\zeta})$$

satisfying (i)–(iii) therein. Observe that for (s, ξ) sufficiently close to (t, ζ) we have

$$S_{s,\xi} \subset\subset N_{t,\zeta},$$

with the distance to the boundary uniformly bounded from below.

Now the family $(h_{s,\xi}) \subset \mathcal{O}(\mathbb{C}^n)$ from Step 2 of the proof of Theorem 1.5, appearing there as (h_t) , depends in our situation in a \mathcal{C}^2 -continuous way on all variables and it suffices to get the same regularity with respect to the parameters of the solutions of the 1st. Cousin Problem with bounds for functions $h_{s,\xi}$ and the coverings $N_{t,\zeta,A} := N_{t,\zeta} \cap A$, $N_{t,\zeta,B} := N_{t,\zeta} \cap B$ of $N_{t,\zeta}$. This we reach by observing that utilizing in the proof of Theorem VII.6.3 from [19] the operator $T_1^{V^{t,\zeta}, V_0^{t,\zeta}}$ instead of \hat{S}_1 (cf. [19, Theorem VII.5.6]) gives, for (s, ξ) close to (t, ζ) , the functions $h_{s,\xi}^{t,\zeta,A} \in \mathcal{O}(N_{t,\zeta,A})$, $h_{s,\xi}^{t,\zeta,B} \in \mathcal{O}(N_{t,\zeta,B})$, depending in a \mathcal{C}^2 -continuous way on all variables, and such that $h_{s,\xi} = h_{s,\xi}^{t,\zeta,A} - h_{s,\xi}^{t,\zeta,B}$ on $N_{t,\zeta} \cap A \cap B$ and with

$$\|h_{s,\xi}^{t,\zeta,A}\|_{N_{t,\zeta,A}} < E\varepsilon, \|h_{s,\xi}^{t,\zeta,B}\|_{N_{t,\zeta,B}} < E\varepsilon, \quad (4.3)$$

where the positive constant E is independent of s and ξ . Note that we have used (iii) from Theorem V.2.5 in [19] (to get suitable regularity with respect to the parameters) and estimates from the beginning of Section V.3.2, also in [19] [to get estimates (4.3)]. Observe it is crucial that the functions $h_{s,\xi}$ are entire.

Then, by the compactness argument, we find W_1, \dots, W_q , and open cover of a neighbourhood of P such that for each $j = 1, \dots, q$ there exist a strictly pseudoconvex and smoothly bounded domain N_j with the property that for all $(s, \xi) \in W_j$ we have $S_{s,\xi} \subset N_j$ with the distance to the boundary uniformly bounded from below, and there

exist functions $h_{s,\xi}^{j,A} \in \mathcal{O}(N_j \cap A)$, $h_{s,\xi}^{j,B} \in \mathcal{O}(N_j \cap B)$, depending in a \mathcal{C}^2 -continuous way on all variables, and such that $h_{s,\xi} = h_{s,\xi}^{j,A} - h_{s,\xi}^{j,B}$ on $N_j \cap A \cap B$ and with

$$\|h_{s,\xi}^{j,A}\|_{N_j \cap A} < E_j \varepsilon, \quad \|h_{s,\xi}^{j,B}\|_{N_j \cap B} < E_j \varepsilon,$$

with positive constant E_j good for all $(s, \xi) \in W_j$.

Let $(p_j)_{j=1}^q$ be a partition of unity subordinated to the covering $(W_j)_{j=1}^q$ of a neighbourhood of P . Define for $(t, \zeta) \in P$

$$h_{t,\zeta}^A := \sum_{j=1}^q p_j(t, \zeta) h_{t,\zeta}^{j,A}, \quad h_{t,\zeta}^B := \sum_{j=1}^q p_j(t, \zeta) h_{t,\zeta}^{j,B}.$$

Then $h_{t,\zeta}^A \in \mathcal{O}(Z_{t,\zeta} \cap A)$, $h_{t,\zeta}^B \in \mathcal{O}(Z_{t,\zeta} \cap B)$, where $Z_{t,\zeta}$ is a neighbourhood of $S_{t,\zeta}$, \mathcal{C}^2 -continuously dependent on (t, ζ) and with the distance to the boundary uniformly (in (t, ζ)) bounded from below. Moreover,

$$\|h_{t,\zeta}^A\|_{U_{t,\zeta} \cap A} < E' \varepsilon, \quad \|h_{t,\zeta}^B\|_{U_{t,\zeta} \cap B} < E' \varepsilon,$$

where positive constant E' does not depend on (t, ζ) . Also,

$$h_{t,\zeta} = h_{t,\zeta}^A - h_{t,\zeta}^B$$

on $Z_{t,\zeta} \cap A \cap B$ and we finish the proof along the lines of the proof of Theorem 1.5.

Finally we may add a family of small corrections to get interpolation at se_1 , which now depends continuously on all variables.

Injectivity in neighbourhoods of $\overline{D_{t,\zeta}^{(\kappa)}} \cup (e_1[0, s])$ with uniform distance to the boundary. This is a consequence of suitably modified techniques presented in Lemma 2.3 from [3]. Namely, let us fix $(t, \zeta) \in P$ and consider the restriction of the function $g_{t,\zeta}$ to the domain $U_{t,\zeta}^0$ with

$$\overline{D_{t,\zeta}^{(\kappa)}} \subset \subset U_{t,\zeta}^0 \subset \subset U_{t,\zeta},$$

created by attaching to $H_{t,\zeta}$ a tubular neighbourhood of radius $\frac{\beta}{4}$ around $e_1[0, s]$. Then for (s, ξ) sufficiently close to (t, ζ) we have

$$\overline{D_{s,\xi}^{(\kappa(s,\xi))}} \subset \subset U_{t,\zeta}^0$$

as well as

$$\overline{D_{t,\zeta}^{(\kappa)}} \subset \subset U_{s,\xi}^0.$$

We claim that for every (s, ξ) sufficiently close to (t, ζ) there exists $W_{s,\xi}$, a neighbourhood of $\overline{D_{s,\xi}^{(\kappa(s,\xi))}} \cup (e_1[0, s])$, such that the distance to the boundary is uniformly bounded from below and with the property that the functions $f_{s,\xi}$ given by Theorem 1.5 for $g_{s,\xi}$ with $\varepsilon < \varepsilon_0$ for some sufficiently small $\varepsilon_0 > 0$ and with the modifications described in the preceding paragraph are all injections on the domains $W_{s,\xi}$. Indeed, let $\varepsilon_0 > 0$ and for (s, ξ) close to (t, ζ) take the decreasing sequences of domains $(U_{(s,\xi),k})_{k \in \mathbb{N}}$ created in a similar way as $U_{s,\xi}^0$, only with the radius of the used tube less than $\frac{1}{k}$ and with $H_{s,\xi}$ replaced by $\overline{D_{s,\xi}^{(\kappa(s,\xi))}}^{\delta + \frac{1}{k}}$ with sufficiently small positive δ (so that it is compactly contained in $H_{s,\xi}$ for large k).

Suppose that for any $k \in \mathbb{N}$ and any $r \in \mathbb{N}$ large enough to ensure $U_{(s,\xi),k} \subset Z_{s,\xi} \cap Z_{t,\zeta}$ for (s, ξ) with the distance to (t, ζ) smaller than $\frac{1}{r}$ (recall $Z_{s,\xi}$ are in place of Ω in Theorem 1.5—see also preceding paragraph—and they do not depend on ε), and any $j_0 \in \mathbb{N}$ there are $\mathbb{N} \ni j \geq j_0$ and $(t_{k,j,r}, \zeta_{k,j,r}) \in P$ with the distance to (t, ζ) smaller than $\frac{1}{r}$, and $a_{k,j,r} \neq b_{k,j,r} \in U_{(t_{k,j,r}, \zeta_{k,j,r}),k}$ such that $f_{t_{k,j,r}, \zeta_{k,j,r}}^j(a_{k,j,r}) = f_{t_{k,j,r}, \zeta_{k,j,r}}^j(b_{k,j,r})$, where the functions $f_{s,\xi}^j$ are given by Theorem 1.5 with $\varepsilon = \frac{1}{j}$ (after modifications described in the preceding paragraph concerning the regularity with respect to parameters and the interpolation condition). We may assume that $(t_{k,j,r}, \zeta_{k,j,r}) \rightarrow (t, \zeta)$ and $a_{k,j,r} \rightarrow a, b_{k,j,r} \rightarrow b$, where $a, b \in \overline{D_{t,\zeta}^{(\kappa(t,\zeta))}} \cup (e_1[0, s])$ as $k, j_0, r \rightarrow \infty$.

Using the injectivity of $g_{t,\zeta}$ and the fact that, by the construction, the family $(f_{s,\xi}^j)$ is uniformly bounded (and hence equicontinuous) near a and near b for (s, ξ) sufficiently close to (t, ζ) , we conclude that in fact it has to be $a = b$. From the methods used in the proof of Lemma 2.3 in [3] we get a may only be an element of the segment $e_1(\delta, s]$. Now, using the properties of the domains $U_{(s,\xi),k}$, the equicontinuity of the family $(f_{t,\zeta}^j)_{(t,\zeta) \in P, j \in \mathbb{N}}$ near a , and performing the computations similar to those from the proof of Lemma 2.3 in [3], we get $f_{t_{k,j,r}, \zeta_{k,j,r}}^j(a_{k,j,r}) \neq f_{t_{k,j,r}, \zeta_{k,j,r}}^j(b_{k,j,r})$ for large k, j, r , a contradiction.

Therefore, for (s, ξ) close to (t, ζ) there exist neighbourhoods $W_{s,\xi}$ of $\overline{D_{s,\xi}^{(\kappa(s,\xi))}} \cup (e_1[0, s])$, such that the distance to the boundary is uniformly bounded from below and with the property that the functions $f_{s,\xi}$ given by Theorem 1.5 for $g_{s,\xi}$ with $\varepsilon < \varepsilon_0$ with sufficiently small $\varepsilon_0 > 0$, and with the modifications described in the preceding paragraph are all injections on the domains $W_{s,\xi}$. Using the compactness argument, we see that for arbitrarily small ε and for all $(t, \zeta) \in P$ there exist neighbourhoods $W_{t,\zeta}$ of $\overline{D_{t,\zeta}^{(\kappa)}} \cup (e_1[0, s])$, such that the distance to the boundary is uniformly bounded from below and with the property that the functions $f_{t,\zeta}$ given by Theorem 1.5 for $g_{t,\zeta}$ with ε are injections on the domains $W_{t,\zeta}$. Moreover, for ε sufficiently small, the domains $W_{t,\zeta}$ do not depend on ε .

To summarize: we proved the existence of a continuous family $(f_{t,\zeta})_{(t,\zeta) \in P}$ of holomorphic embeddings $\overline{D_{t,\zeta}^{(\kappa)}} \cup (e_1[0, s]) \rightarrow \mathbb{C}^n$ uniformly (in all variables) close to $\Theta_{t,\zeta}(l_{t,\zeta}(\gamma_{t,\zeta}[0, 1]))$ and with the property that $f_{t,\zeta}(e_1[0, s])$ is perpendicular to

$(\Theta_{t,\zeta} \circ l_{t,\zeta})(\mathbb{S}^{2n-1}(R))$. Note that in the case of a single domain treated in [3], similar result is obtained by different methods, based on [14].

Let

$$\hat{h}_{t,\zeta} := (\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1} \circ f_{t,\zeta} \circ (\Theta_{t,\zeta} \circ l_{t,\zeta}),$$

on the set where the composition is defined. In the last part of the proof we shall construct, for sufficiently small positive s , the continuous family $(F_{t,\zeta})_{(t,\zeta) \in P}$ of holomorphic embeddings $\overline{G_t^{(\kappa)}}$ such that for all $(t, \zeta) \in P$ we have

- (1') $F_{t,\zeta}(\zeta) = (\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1}(se_1)$,
- (2') $F_{t,\zeta}(\overline{G_t^{(\kappa)}} \cap Y_{t,\zeta}) \subset (\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1}((e_1[0, s - 2b])^c \cup \mathbb{B}(e_1(s - b), b) \cup \{se_1\})$,
- (3') $\|F_{t,\zeta} - \text{Id}\|_{\overline{G_t^{(\kappa)}} \setminus Y_{t,\zeta}}$ is arbitrarily small,

with some small positive s' , b , c , where $Y_{t,\zeta}$ is some small neighbourhood of ζ . Taking, for all $t \in \sigma\mathbb{D}$ and $\zeta \in \partial G_t$, the composition

$$h_{t,\zeta} := \hat{h}_{t,\zeta} \circ F_{t,\zeta}$$

will end the proof.

Construction of the family $(F_{t,\zeta})_{(t,\zeta) \in P}$. Observe that there exist $r, S > 0$ such that for all $(t, \zeta) \in P$ we have

$$\Theta_{t,\zeta}(l_{t,\zeta}(\overline{(G_t^{(\kappa)} \cap \mathbb{B}(\zeta, r)) \setminus \{\zeta\}})) \subset \mathbb{B}(-Se_1, S)$$

and

$$\Theta_{t,\zeta}(l_{t,\zeta}(\zeta)) = 0 \in \partial \mathbb{B}(-Se_1, S).$$

By Theorem 3.1 from [3], for all sufficiently small positive s, δ, c , and b with $b < s, c < s$ there exists a holomorphic injection $\Phi : \mathbb{B}(-Se_1, S) \rightarrow \mathbb{C}^n$ such that

- (1+) $\|\Phi - \text{Id}\|_{\mathbb{B}(-Se_1, S) \setminus \mathbb{B}(0, \delta)}$ is arbitrarily small,
- (2+) $\Phi(0) = se_1$,
- (3+) $\Phi(\mathbb{B}(-Se_1, S) \cap \mathbb{B}(0, \delta)) \subset (e_1[0, s - 2b])^c \cup \mathbb{B}(e_1(s - b), b) \cup \{se_1\}$.

Define, for $(t, \zeta) \in P$,

$$\tilde{F}_{t,\zeta}(z) := ((\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1} \circ \Phi \circ (\Theta_{t,\zeta} \circ l_{t,\zeta}))(z)$$

for z from $\mathbb{B}(\zeta, r)$ intersected with some neighbourhood of $\overline{G_t^{(\kappa)}}$ of uniform size (in t, ζ). Then, after eventually shrinking the domains of definition, the family $(\tilde{F}_{t,\zeta})_{(t,\zeta) \in P}$ depends \mathcal{C}^2 -continuously on all variables and for every $(t, \zeta) \in P$ we have

- (1*) $\tilde{F}_{t,\zeta}$ is a holomorphic injection,

$$(2^*) \quad \tilde{F}_{t,\zeta}(\zeta) = (\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1}(se_1),$$

$$(3^*) \quad \tilde{F}_{t,\zeta}(\overline{G_t^{(\kappa)}} \cap \mathbb{B}(\zeta, s')) \subset (\Theta_{t,\zeta} \circ l_{t,\zeta})^{-1}((e_1[0, s-2b])^c \cup \mathbb{B}(e_1(s-b), b) \cup \{se_1\})$$

with some sufficiently small $s' \in (0, s)$.

Let us consider, for sufficiently small positive k , the family of Cartan pairs $((A_{t,\zeta}, B_{t,\zeta}))_{(t,\zeta) \in P}$, where

$$A_{t,\zeta} := \overline{G_t^{(\kappa)}} \cap \mathbb{B}(\zeta, k), \quad B_{t,\zeta} := \overline{G_t^{(\kappa)}} \setminus \mathbb{B}\left(\zeta, \frac{k}{2}\right).$$

Define

$$C_{t,\zeta} := A_{t,\zeta} \cap B_{t,\zeta}$$

and observe that if k is small enough, then we have

$$\Theta_{t,\zeta}(l_{t,\zeta}(C_{t,\zeta})) \subset \subset \mathbb{B}(-se_1, S)$$

with the distance to the boundary uniformly (in t, ζ) bounded from below. If in the choice of Φ the constant δ [and consequently also s' from (4*)] was sufficiently small, then for all $(t, \zeta) \in P$ the mappings $\tilde{F}_{t,\zeta}$ are uniformly arbitrarily close to Id in neighbourhoods of $C_{t,\zeta}$ of uniform (in t, ζ) size. By Theorem 2.4 and Remark 2.5, we get the existence of the continuous family $(\alpha_{t,\zeta})_{(t,\zeta) \in P}$ of biholomorphic mappings in neighbourhoods of $A_{t,\zeta}$, of uniform (in t, ζ) size, interpolating the identity to arbitrarily high order at ζ , and the continuous family $(\beta_{t,\zeta})_{(t,\zeta) \in P}$ of biholomorphic mappings in neighbourhoods of $B_{t,\zeta}$ of uniform (in t, ζ) size, such that the family

$$F_{t,\zeta} := \begin{cases} \tilde{F}_{t,\zeta} \circ \alpha_{t,\zeta}, & \text{in a neighbourhood of } A_{t,\zeta} \\ \beta_{t,\zeta}, & \text{in a neighbourhood of } B_{t,\zeta} \end{cases}, \quad (t, \zeta) \in P,$$

fulfilling (1')–(3'), is the last piece of our puzzle. \square

5 Strongly Linearly Convex Case

Example 5.1 Let $k \geq 3$ and let ρ and G_t be as in Problem 1.2 and assume additionally that G_t is strongly linearly convex for each t . Let $\sigma < \sigma' \in (0, 1)$.

By Proposition 2.2.3 in [12], for any t there exist U_t , a neighbourhood of ∂G_t and a \mathcal{C}^{k-1} -continuous mapping $\pi_t : U_t \rightarrow \partial G_t$ such that for $x \in U_t$, $\pi_t(x)$ is a unique point from ∂G_t that realizes $\text{dist}(x, \partial G_t)$. By analysing the proof of that proposition, we see that U_t and π_t may be chosen to be \mathcal{C}^{k-1} -continuously dependent on t . Moreover, by our assumptions, the choice may be carried out in such a way that for s, t close enough we have $\partial G_t \subset U_s$ and $\partial G_s \subset U_t$.

For each t let us choose open sets

$$\partial G_t \subset U_t'' \subset \subset U_t' \subset \subset U_t,$$

varying in a C^{k-1} -continuous manner with t , and cutoff functions χ_t such that $\chi_t = 0$ on $\mathbb{C}^n \setminus U'_t$ and $\chi_t = 1$ on U''_t , also varying in a C^{k-1} -continuous way.

If now s, t are close enough, the mapping

$$\varphi_{st} : \mathbb{C}^n \ni z \rightarrow z + (\pi_s(z) - \pi_t(z))\chi_s(z)\chi_t(z) \in \mathbb{C}^n$$

is a C^{k-1} -diffeomorphism (cf. [22], p. 400). Obviously $\varphi_{st}(\partial G_t) \subset \partial G_s$, and even an equality must hold there, because boundary of strongly linearly convex domain of class at least C^2 is diffeomorphic with \mathbb{S}^{2n-1} (see [23]), and \mathbb{S}^{2n-1} is not diffeomorphic with any of its proper subsets. Therefore, for s, t close enough φ_{st} constitutes a C^{k-1} -diffeomorphism between ∂G_t and ∂G_s (and indeed between \overline{G}_t and \overline{G}_s).

We would like to construct a C^{k-1} -continuous family of C^{k-1} -diffeomorphisms $(\psi_t : \mathbb{C}^n \rightarrow \mathbb{C}^n)_{t \in \sigma' \overline{\mathbb{D}}}$, mapping \overline{G}_t diffeomorphically to \overline{G}_0 . Define

$$\mathcal{R} := \{r \in [0, \sigma'] : \text{there exists such a family for } t \in r \overline{\mathbb{D}}\}.$$

Obviously $\mathcal{R} \neq \emptyset$. Furthermore, \mathcal{R} is open in $[0, \sigma']$: let $r_0 \in \mathcal{R}$ and let $(\tilde{\psi}_t)_{t \in r_0 \overline{\mathbb{D}}}$ be a suitable family. In virtue of the observation we just made, that for s, t close enough φ_{st} is a diffeomorphism between \overline{G}_t and \overline{G}_0 , we may, without loss of generality, assume that $r_0 \neq 0$. Similarly, we may assume $r_0 \neq \sigma'$. It is apparent that we only need to show that there exists some $r \in (r_0, \sigma')$ and the family $(\psi_t)_{t \in r \overline{\mathbb{D}}}$ of required diffeomorphisms. Let us consider the covering of the circle $|t| = r_0$ by the finite family U_1, \dots, U_m of closed balls, with the multiplicity of the covering equal 2, such that for any $j \in \{1, \dots, m\}$, if we denote by a_j the centre of the ball U_j , then $|a_j| < r_0$ and $a_j \notin U_k$ for every $k \in \{1, \dots, j-1, j+1, \dots, m\}$, and moreover, for every $j \in \{1, \dots, m\}$ and for every $s, t \in U_j$, φ_{st} is a diffeomorphism between \overline{G}_t and \overline{G}_s . Put

$$\psi_t := \begin{cases} \tilde{\psi}_t, & t \in r_0 \overline{\mathbb{D}} \\ \tilde{\psi}_{d(t)} \circ \varphi_{d(t)t}, & t \in \bigcup_{j=1}^m U_j \setminus r_0 \overline{\mathbb{D}}, \end{cases}$$

where $d(t)$ denotes a point from $|t| = r_0$ closest to t . Since we take t outside $r_0 \overline{\mathbb{D}}$, we get $d(t)$ is unique, and the function d is smooth. Then $(\psi_t)_{t \in (\bigcup_{j=1}^m U_j) \cup r_0 \overline{\mathbb{D}}}$ is a C^{k-1} family of required diffeomorphism, which ends the proof of openness of \mathcal{R} (after restricting the set of parameters to $r \overline{\mathbb{D}}$ with suitable $r \in (r_0, \sigma')$).

\mathcal{R} is also closed: let $r_v \rightarrow r_0$. Then we cover the circle $|t| = r_0$ with balls U_1, \dots, U_m as in the proof of the openness, and we observe that there exists a v_0 with $\{|t| = r_{v_0}\} \subset \bigcup_{j=1}^m U_j$ and $|a_j| < r_{v_0}$, $j = \{1, \dots, m\}$. Now we use a similar argument as in the proof of openness to produce a suitable family of diffeomorphisms for $t \in r_0 \overline{\mathbb{D}} \subset (\bigcup_{j=1}^m U_j) \cup r_{v_0} \overline{\mathbb{D}}$.

Let $\Phi : \overline{G}_0 \rightarrow \overline{I}$ be a C^1 -diffeomorphism, where I is a complete circular domain with boundary of class C^2 (see [18]) and let $\Psi : \overline{I} \rightarrow \overline{\mathbb{B}}$ be a C^1 -diffeomorphism, whose existence was pointed out to the author by Andrea Spiro (private communication; the proof requires some modifications of standard proofs that open star-shaped

domain is diffeomorphic to the unit ball, cf. [10]). The composition of these latter diffeomorphisms gives a \mathcal{C}^1 -diffeomorphism between $\overline{G_0}$ and $\overline{\mathbb{B}}$, and by the argument as in the proof of Theorem 5.2 in [3], there exists Θ , a \mathcal{C}^1 -diffeomorphism of \mathbb{C}^n such that $\Theta(\overline{G_0}) = \overline{\mathbb{B}}$. Define $\Gamma_t := \Theta \circ \psi_t$, which is a \mathcal{C}^{k-1} -continuous family of \mathcal{C}^1 -diffeomorphisms of \mathbb{C}^n with $\Gamma_t(\overline{G_t}) = \overline{\mathbb{B}}$. Also, if $R > 0$ is large enough, all the mappings ψ_t are equal Id on the preimage by Θ of the sphere $\|z\| = R$. Now the family of embedded curves $(\gamma_{t,\xi})$ as in Theorem 1.3 may be produced as in the proof of Theorem 5.1 from [3] and with the aid of results from Chapter 3 in [11] and the discussion from [1, Section 4.1] concerning the parameter dependence of the evolution operators of parameter-dependent vector fields.

Remark 5.2 Observe that the argument similar to the one just presented would remain true if only we knew at the very beginning that the closures of the domains we work with are all \mathcal{C}^1 -diffeomorphic to the closed unit ball (compare Theorem 5.2 in [3]).

6 Concluding Remarks

As we have observed in the Introduction, it seems that with the existing methods at hand we are not able to omit the assumption about the existence of the family $(\gamma_{t,\xi})$ in Theorem 1.3. On the other hand, even with this additional assumption, the full solution for the question (B) from the Problem 1.2, i.e. passing from “compact” case to the case where the set of parameters equals \mathbb{D} , and increasing the regularity of the family $(h_{t,\xi})$ with respect to all variables (if the domains vary in suitably more regular manner) still requires developing some subtle tools, for example, a qualitatively new version of parameter Forstnerič splitting lemma, where we would have better regularity of the families (α_t) and (β_t) , and we would be able to change the size of μ - and τ -hulls appearing there pretty arbitrarily with the parameter (see Theorem 2.4 for the notation). We hope to undertake this problem in the forthcoming paper(s).

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References

1. Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications. Applied Mathematical Sciences, vol. 75, 1st edn. Springer, New York (1988)
2. Bracci, F., Fornæss, J.E.: The range of holomorphic maps at boundary points. *Math. Ann.* **359**(3–4), 909–927 (2014)
3. Deng, F., Fornæss, J.E., Wold, E.F.: Exposing boundary points of strongly pseudoconvex subvarieties in complex spaces. *Proc. Am. Math. Soc.* **146**(6), 2473–2487 (2018)
4. Deng, F., Guan, Q., Zhang, L.: Properties of squeezing functions and global transformations of bounded domains. *Trans. Am. Math. Soc.* **368**(4), 2679–2696 (2016)

5. Diederich, K., Fornæss, J.E., Wold, E.F.: Exposing points on the boundary of strictly pseudoconvex or a locally convexifiable domain of finite 1-type. *J. Geom. Anal.* **24**, 2124–2134 (2014)
6. Fornæss, J.E., Wold, E.F.: An Estimate for the Squeezing Function and Estimates of Invariant Metrics. *Complex analysis and geometry. Springer Proceedings in Mathematics and Statistics*, vol. 144, pp. 135–147. Springer, Tokyo (2015)
7. Fornæss, J.E., Forstnerič, F., Wold, E.F.: Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan, to appear in the volume. In: *Advancements in Complex Analysis*. Springer, New York. [arXiv:1802.03924](https://arxiv.org/abs/1802.03924), preprint (2018)
8. Forstnerič, F.: Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis. In: *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge/A Series of Modern Surveys in Mathematics*, vol. 56, 2nd edn. Springer, Berlin (2017)
9. Gong, X., Kim, K.-T.: The $\bar{\partial}$ -equation on variable strictly pseudoconvex domains. *Math. Z.* **290**(1–2), 111–144 (2018)
10. Gonnord, S., Tosel, N.: *Calcul Différentiel Thèmes d'Analyse Pour l'Agrégation*. Paris Ellipses DL, Paris (1998)
11. Hirsch, M.W.: *Differential Topology*. Graduate Texts in Mathematics, vol. 33. Springer, New York (1976)
12. Jarnicki, M., Pflug, P.: *Extension of Holomorphic Functions*, de Gruyter Expositions in Mathematics, vol. 34. Walter de Gruyter, Berlin (2000)
13. Krantz, S.G.: *Function Theory of Several Complex Variables*, Reprint of the 1992 Edition. AMS Chelsea Publishing, Providence, RI (2001)
14. Kutzschebauch, F., Wold, E.F.: Carleman approximation by holomorphic automorphisms of \mathbb{C}^n . *J. Reine Angew. Math.* (2015). <https://doi.org/10.1515/crelle-2015-0056>
15. Lewandowski, A.: Families of strictly pseudoconvex domains and peak functions. *J. Geom. Anal.* **28**(3), 2466–2476 (2018)
16. Lewandowski, A.: Strictly pseudoconvex domains and smoothly varying peak functions. *J. Math. Anal. Appl.* **473**, 1073–1080 (2019)
17. Manne, P.E.: Carleman approximation on totally real submanifolds of a complex manifold, In: *Several complex variables (Stockholm, 1987/1988)*, vol. 38 of *Math. Notes*, pp. 519–528. Princeton Univ. Press, Princeton, NJ (1993)
18. Pang, M.-Y.: Smoothness of the Kobayashi metric of non-convex domains. *Int. J. Math.* **4**(6), 953–987 (1993)
19. Range, R.M.: *Holomorphic functions and integral representations in several complex variables*. In: *Graduate Texts in Mathematics*, vol. 108, Springer, New York (1986) (Corrected second printing, 1998)
20. Simon, L.: A splitting lemma for biholomorphic maps on continuously varying domains, Master Thesis, University of Oslo, Oslo (2014)
21. Simon, L.: A parameter version of Forstnerič's splitting lemma. *J. Geom. Anal.* (2018). <https://doi.org/10.1007/s12220-018-0073-8>
22. Stout, E.L.: *Polynomial Convexity*. Progress in Mathematics, vol. 261. Birkhäuser Boston Inc., Boston, MA (2007)
23. Torres, D.M.: A note on strict \mathbb{C} -convexity. *Rev. Mat. Complut.* **25**, 125–137 (2012)
24. Ugolini, R.: A parametric Jet-interpolation theorem for holomorphic automorphisms of \mathbb{C}^n . *J. Geom. Anal.* **27**(4), 2684–2699 (2017)
25. Weickert, B.J.: Automorphisms of \mathbb{C}^n . <http://home.wlu.edu/weickertb/papers/interp.pdf>, preprint
26. Wold, E.F.: Asymptotics of invariant metrics in the normal direction and a new characterisation of the unit disk. *Math. Z.* **288**(3–4), 875–887 (2018)
27. Zhang, L.: Intrinsic derivative, curvature estimates and squeezing function. *Sci. China Math.* **60**(6), 1149–1162 (2017)